

# Quark Mass Deformation of Holographic Massless QCD

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## Abstract

We propose several quark mass deformations of the holographic model of massless QCD using the D4/D8/ $\overline{\text{D8}}$ -brane configuration proposed by Sakai and Sugimoto. The deformations are based on introducing additional D4- or D6-branes away from the QCD D4 branes. The idea is similar to extended technicolor theories, where the chiral symmetry breaking by additional D-branes is mediated to QCD to induce non-zero quark masses. In the D-brane picture as well as the holographic dual gravity description, the quark and the pion masses are generated by novel worldsheet instantons with finite area. We also derive the Gell-Mann-Oakes-Renner relation, and find the value of the chiral condensate in the Sakai-Sugimoto model.

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# 1 Introduction

AdS/CFT correspondence [1] has provided us with a new avenue to compute observables in strongly interacting gauge theories via weakly coupled gravitational descriptions. Even though the correspondence is still a conjecture, many nontrivial computations have been performed and checked to be consistent. One of the most important applications, and also challenges, would be to analyze real QCD in Nature using the AdS/CFT correspondence.

Recently, there proposed a holographic model of QCD by Sakai and Sugimoto, where the non-Abelian chiral symmetry and its spontaneous breaking is geometrically realized via a D4/D8/ $\overline{\text{D8}}$  configuration [2]. The model predictions for the meson spectrum and the couplings between them compare well with experiments with a good accuracy. Baryons in this model are also investigated in [3]. Moreover, this holographic model has an ability of predicting possible glue-ball interactions to mesons as studied in [4]. There are many other QCD observables that have been analyzed in this model, which show that the model is a good approximation to real QCD at least in low energy, large  $N_c$ -limit.

Despite its success in many aspects, this model has an apparent shortcoming that the pions are massless. This is because the quarks are massless from the construction. Therefore one of the important questions is to find a deformation of the theory which corresponds to introducing bare quark masses. Once we identify the deformation, we should be able to show that the Gell-Mann-Oakes-Renner (GOR) relation [5] is satisfied, and moreover, we can compute the value of the chiral condensate of the Sakai-Sugimoto model through this relation. Pion dynamics would be the most important in holographic QCD because the Sakai-Sugimoto model approaches real QCD in the low energy limit, thus the identification of the quark mass deformation in the model is indispensable.

The chiral quarks appear as the lowest massless modes of the open string stretching between intersecting  $N_c$  D4-branes and  $N_f$  D8-branes (or  $\overline{\text{D8}}$ -branes) in the model. Therefore one would naively expect that the quarks become massive if one can realize the situation where the D8 and  $\overline{\text{D8}}$ -branes do not intersect with the D4-branes. Some of the authors [6] studied such local deformations near the D4-branes (which do not deform the asymptotic configuration away from the D4-branes) to realize the pion mass, but the quarks in this set-up are suggested to be still massless. This is in fact consistent with the proposed relation [7] between the classical value of action in the bulk and the partition function in CFT for the AdS<sub>5</sub>/CFT<sub>4</sub> correspondence, since the value of bulk field at the AdS boundary is related with the strength of coupling, such as the mass, in CFT. There are attempts to include D8- $\overline{\text{D8}}$ -brane tachyon in the effective action to explicitly break the chiral symmetry [8]. However, tachyon action is not reliable as there is no consistent truncation of the tachyon theory to low energy. Furthermore, possible relation to the original Sakai-Sugimoto brane configuration is not obvious.

In this work, we propose deformations of the Sakai-Sugimoto model which correspond to the introduction of the quark mass. Our deformations have additional D-branes away from the original confining  $N_c$  D4-branes, which still have a reasonable field theory interpretation.

The original Sakai-Sugimoto model is restored once these additional D-branes are moved to spatial infinity in the bulk. In the field theory view-point, our idea is rather similar to the technicolor model [9], so our construction can be thought of as a holographic realization of technicolor model. In the minimal technicolor model, we have an additional sector to the original QCD, which contains new techni-quarks interacting via new technicolor interactions. The quarks/techni-quarks are massless at UV, but the chiral symmetry (the electroweak symmetry in this case) is spontaneously broken by a techni-quark condensate driven by the strong technicolor gauge interactions. To realize QCD quark mass from this breaking, we need to introduce extended technicolor interactions between quarks and techni-quarks, in addition to the usual weak interactions [10]. This setup is actually close to the holographic dual of extended type technicolor model by applying the D-brane configuration of Sakai-Sugimoto model in [11]. This extended gauge group can often be realized in GUT type construction, where quarks and techni-quarks are in a same multiplet, and the GUT gauge symmetry is broken to QCD and technicolor at some high scale. Through off-diagonal massive gauge bosons, QCD quarks get explicit bare mass term from the techni-quark condensate. From the view-point of QCD, we have a realization of explicit chiral symmetry breaking mass for the quarks.

To realize this idea in our set-up, we introduce additional  $N'$  D4-branes (we call them D4'-branes) which are parallel to, but separated from the original  $N_c$  D4-branes of QCD gauge symmetry. The GUT gauge symmetry is  $SU(N_c + N')$  when D4'-branes are on top of the original D4-branes, and it is broken to  $SU(N_c) \times SU(N')$  by a Higgs mechanism via separating D4' from D4. Massive off-diagonal gauge bosons appear as strings between D4 and D4'. We add  $N_f$  D8/ $\overline{\text{D8}}$ -branes as in [2] to introduce massless quarks and techni-quarks, which are charged under the chiral symmetry  $U(N_f) \times U(N_f)$ . By assuming a strong technicolor  $SU(N')$  dynamics on the D4'-branes, which replaces D4'-branes with the Witten geometry, the chiral symmetry is spontaneously broken by adjoining D8/ $\overline{\text{D8}}$ -branes there, as in the Sakai-Sugimoto model but with technicolor instead of QCD. This breaking will be mediated to the QCD sector of  $N_c$  D4-branes by, for example, the massive gauge bosons coming from open strings between the D4 and D4'-branes. This is expected to induce the bare masses for the QCD quarks.

From the Feynman graphs that would induce quark masses from a techni-quark condensate, we can easily identify the corresponding string worldsheets that mediate this phenomenon in our D-brane configuration in flat space. We consider a closed loop surrounded by D4-D8-D4'- $\overline{\text{D8}}$ -branes (see Fig.5) and a disk *worldsheet instanton* whose boundary is ending on this closed loop. It is responsible for the extended technicolor interactions coupling quarks to techni-quarks. As we go to the strongly coupled field theory, or weakly coupled gravity regime, we keep track of how this worldsheet looks like in the gravity description, as we will study in Sec. 2. In the gravity dual description, where both D4 and D4'-branes are replaced by the near horizon geometry of multi center solution, we have a closed loop with the probe D8-branes only, since D8 and  $\overline{\text{D8}}$ -branes are smoothly connected at two throats created by the D4 and the D4'-branes (see Fig.7). The previous worldsheet instanton in the weak coupling

picture is now ending on this closed loop, and is understood as a leading order contribution, since this is a planar diagram with one boundary in large  $N$  gauge theory [12]. We will show that this worldsheet instanton amplitude indeed induces the lowest mass perturbation that one may expect in the low energy chiral Lagrangian of pions. We will also show that the GOR relation is satisfied. This will be presented in detail in Sec. 2.

Since the chiral symmetry is broken spontaneously in view of the whole GUT theory, there still exist massless Goldstone bosons. These modes will be localized around D4' throat when the techni-quark condensate scale is much bigger than the QCD scale. In the gravity picture, they would correspond to the Wilson line on D8 from the asymptotic boundary to the D4' throat and again to the boundary. We have to take a limit to decouple these Nambu-Goldstone bosons by scaling  $N' \rightarrow \infty$  first and putting D4'-branes at far UV region from the D4-branes, while keeping finite the mass of pions associated with the chiral symmetry breaking realized by  $N_c$  D4-branes. Also, it seems complicated to analyze multi-center solutions of D4-branes explicitly. It is difficult to realize flavor dependent quark mass in this set-up, too.

We then consider introducing  $N'$  D6-branes instead of the D4'-branes, which is free of these difficulties. This deformation keeps the essential idea of introducing  $N'$  D4-branes, but has great advantages of computability without taking any subtle limits. Therefore, we can compute the quark mass as well as the pion mass more rigorously, and consequently we can estimate the value of the chiral condensate of the Sakai-Sugimoto model via the GOR relation. We will also see that the quark masses derived from our results with the pion mass as an input will be around 6 MeV, which happens to be quite close to the real QCD. The numerical value of the computed chiral condensate is close to the results of lattice QCD. This will be presented in Sec. 3.

In the next section, we first give the field theoretical interpretation of introducing the  $N'$  D4-branes and describe the Feynman graphs which induce the quark masses. Then we study the corresponding string worldsheet instantons both in the D-brane configuration in flat space and in the holographic dual gravity description, and estimate the quark and pion masses. After explaining difficulties of precise computations, we instead introduce  $N'$  D6-branes and study the corresponding interpretation in QCD side in Sec. 3, where we carry out explicit computations of quark and pion masses. We show in detail that the up and down quarks have masses around 6 MeV in our model. It is important to stress that we can also realize flavor dependent quark masses with our D6-branes. Since our idea has a field theoretical interpretation which can be applied to a wide range of holographic QCD models, we apply our idea to a holographic approach of QCD proposed by Kruczenski *et.al* [13] in Sec.4. We give a detail about how our idea works in their model. In Sec. 5, we conclude with discussions.

## 2 Quark mass deformation and worldsheet instantons

In the Sakai-Sugimoto model with  $N_c$  D4,  $N_f$  D8 and  $N_f$   $\overline{\text{D8}}$ -branes, QCD gauge bosons are realized on the D4-branes and the chiral flavor symmetry  $U(N_f)_L \times U(N_f)_R$  is realized as the

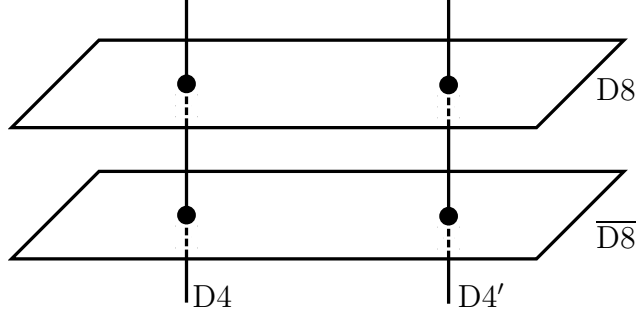


Figure 1: The D-brane configuration

gauge symmetry on the  $N_f$  D8 and  $N_f$   $\overline{\text{D8}}$ -branes. The left (right) handed massless quarks are localized at the intersection between the D4-branes and the D8- ( $\overline{\text{D8}}$ -) branes. In the holographic dual description, the near horizon geometry of the D4-branes develops a throat which naturally explains confinement. The chiral symmetry breaking in QCD is nicely realized since the D8 and  $\overline{\text{D8}}$ -branes are connected smoothly at the throat.

If we introduce additional D4-branes (D4'-branes) which are parallel to but separated from the original D4-branes, the chiral symmetry is also broken by the D4'-branes in the holographic dual description of strong technicolor dynamics on the D4'-branes. Since there are massive gauge bosons from the open strings stretching between the D4 and D4'-branes, the effect of the chiral symmetry breaking at the D4'-branes will be mediated into QCD sector on the D4-branes and the quark mass terms are resultantly induced. We will pursue this idea in this section.

## 2.1 Quark mass deformation

We first describe the D-brane configuration and study the low energy theory on the D-branes. The D-brane configuration which we now consider consists of D4, D8 and  $\overline{\text{D8}}$ -branes (see Fig.1) and is summarized in the table:

	0	1	2	3	4	5	6	7	8	9
$N_c$ D4	○	○	○	○	○					
$N_f$ D8/ $\overline{\text{D8}}$	○	○	○	○		○	○	○	○	○
$N'$ D4'	○	○	○	○	○					

where  $N_c$  D4-branes are localized at  $x^5 = \dots = x^9 = 0$  and  $N'$  D4-branes are localized at  $x^5 \neq 0$  and  $x^6 = \dots = x^9 = 0$ . The D8 and  $\overline{\text{D8}}$ -branes are localized at  $x^4 = 0$  and  $x^4 = \delta\tau/2$  where  $x^4$  direction is compactified by a supersymmetry breaking  $S^1$  with the periodicity  $x^4 \sim x^4 + \delta\tau$ . The compactification scale determines the scale of this system and  $M_{KK} \equiv 2\pi/\delta\tau \sim 1$  GeV in the Sakai-Sugimoto model [2].

The  $N'$  D4-branes realize another massless QCD with  $SU(N')$  gauge symmetry as a low energy effective theory in addition to the massless QCD with  $SU(N_c)$  in the Sakai-Sugimoto

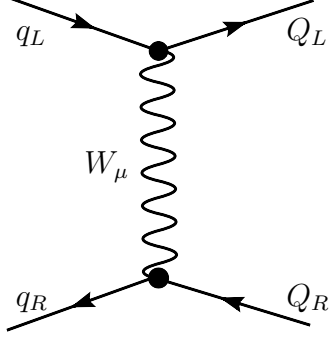


Figure 2: The four fermi interaction mediated by massive gauge boson  $W_\mu$ .

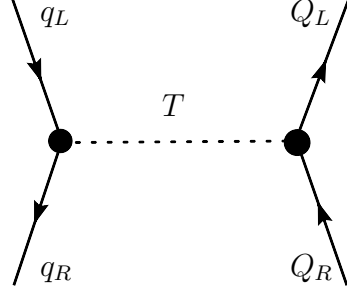


Figure 3: The four fermi interaction mediated by massive scalar field  $T$ .

model. Thus the massless modes are  $SU(N_c) \times SU(N')$  gauge bosons and the two sets of left handed and right handed chiral massless quarks  $(q_L, q_R)$  and  $(Q_L, Q_R)$  which are fundamentally charged under  $SU(N_c)$  and  $SU(N')$  respectively. The massless modes are summarized in the table:

	$SU(N_c)$	$SU(N')$	$U(N_f)_L$	$U(N_f)_R$
$A_\mu$	adj.			
$q_L$	$\square$		$\square$	
$q_R$	$\square$			$\square$
$A'_\mu$		adj.		
$Q_L$		$\square$	$\square$	
$Q_R$		$\square$		$\square$

Since  $Q_L$  ( $Q_R$ ) come from the open strings stretching between  $N'$  D4 and  $N_f$  D8 ( $\overline{\text{D8}}$ )-branes, the flavor symmetry is still  $U(N_f)_L \times U(N_f)_R$  and there are massive modes which connect  $(q_L, q_R)$  and  $(Q_L, Q_R)$ . They are the modes originated from the open strings stretching between the D4 and D4'-branes or D8 and  $\overline{\text{D8}}$ -branes. The lightest modes from these two strings are massive gauge bosons  $W_\mu$  and complex scalar fields  $T$  whose charges are summarized in the table:

	$SU(N_c)$	$SU(N')$	$U(N_f)_L$	$U(N_f)_R$
$W_\mu$	$\square$	$\square$		
$T$			$\square$	$\square$

Their masses are computed from the distance between the D4- and D4'-branes or the D8- and the  $\overline{\text{D8}}$ -branes.<sup>¶</sup> The quarks have a gauge and Yukawa interaction with  $W_\mu$  and  $T$  fields and then there are two Feynman graphs in Fig.2-3 which generate the four fermi interactions  $\bar{q}_L q_R \bar{Q}_L Q_R + \text{h.c.}$  mediated by the massive gauge or the scalar fields.

<sup>¶</sup> The field  $T$  has been referred to as a “tachyon” since it is from a string connecting the D8 and the  $\overline{\text{D8}}$ -branes. However, when those branes are separated as in the present case, it is massive. In the low energy limit  $l_s \rightarrow 0$  on the D4-branes, the mass of  $T$  diverges.

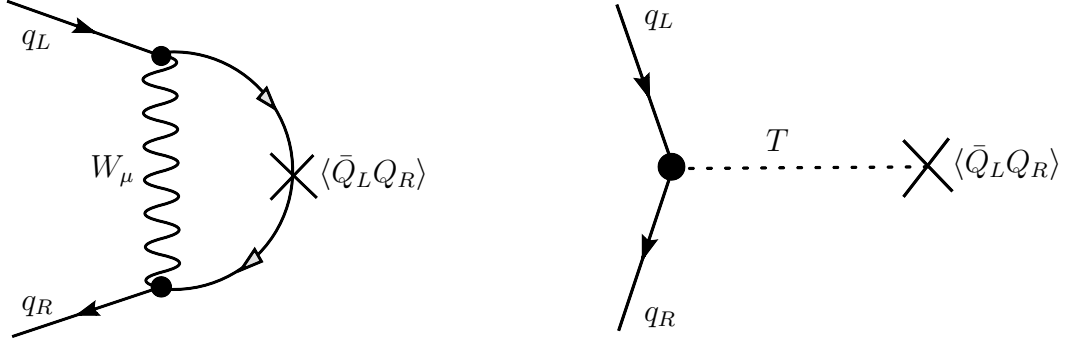


Figure 4: The quark mass is generated from the condensate  $\langle \bar{Q}_L Q_R \rangle$ .

At the low energy both  $SU(N_c)$  and  $SU(N')$  gauge symmetries become confined and the chiral symmetry is broken by the condensate  $\langle \bar{q}_L q_R \rangle$  and  $\langle \bar{Q}_L Q_R \rangle$ . Then by picking up the condensate  $\langle \bar{Q}_L Q_R \rangle$ , the quark  $(q_L, q_R)$  becomes massive  $m_q \propto \langle \bar{Q}_L Q_R \rangle$ . See Fig. 4. This mechanism is quite similar to the mechanism of inducing the quark masses in extended technicolor theories.

From the low energy effective theory on D-branes in the weak coupling picture, we can estimate the quark masses  $m_q$ . The effective theory on the D4-branes is essentially a five dimensional theory and quarks are localized at spatially separated 4 dimensional surfaces where they intersect with the D8-branes. Thus the calculation involves Kaluza-Klein (KK) reduction along the  $S^1$  with a few steps and a similar calculation has been done in a context of brane world scenario in [14] (see appendix A for the calculations). We just mention that the amplitude in Fig.2 is suppressed by  $\exp[-\pi M_W/M_{KK}]$  since the W-boson with the mass  $M_W$  must propagate from  $\tau = 0$  to  $\tau = \delta\tau/2 = \pi/M_{KK}$ . We can similarly compute the Feynman graph Fig.3 and obtain the same suppression  $\exp[-\pi M_W/M_{KK}]$ : the D8/ $\overline{D8}$ -brane tachyon  $T$  with mass  $\frac{1}{2\pi\alpha'} \frac{\delta\tau}{2}$  propagating horizontally from the D4' to the D4 by distance  $U_0 = 2\pi\alpha' M_W$ . In string theory, the string worldsheet which represents these two graphs is identical. It is  $st$ -channel duality on the string worldsheet which manifests in these two Feynman graphs. Then it is in fact easier to estimate the masses by studying the corresponding string worldsheets.

## 2.2 Worldsheet instanton

The string worldsheets which correspond to the Feynman graphs in Fig.2-3 can be easily identified. The chiral quarks are localized on the intersections of D-branes of different types and open strings stretching between D-branes of the same type induce massive gauge or scalar fields. Also the graphs in Fig.2-3 are tree graphs and we conclude that the corresponding string worldsheets are disk worldsheets whose boundary are on the closed loop surrounded by D4-D8-D4'- $\overline{D8}$  branes and which have the four boundary twist vertex operators at the intersections of the D-branes. Since these worldsheets are localized in time direction, these

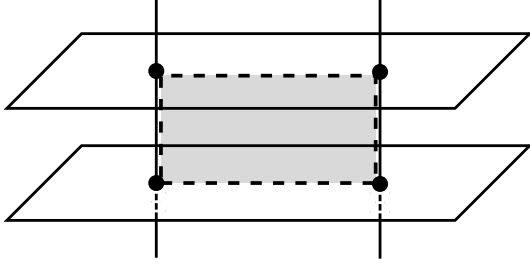


Figure 5: The shaded region denotes a worldsheet instanton. A dotted line is a closed loop surrounded by the D4-D8-D4'- $\overline{\text{D8}}$  branes.

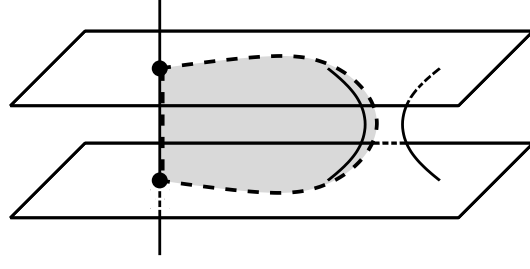


Figure 6: The shaded region denotes a worldsheet instanton. The throat is developed at the location of D4'-branes in the geometry.

are worldsheet instantons which are denoted by a shaded region in Fig.5. <sup>||</sup>

The quark masses are induced by the chiral condensate  $\langle \bar{Q}_L Q_R \rangle$  realized by strongly interacting  $SU(N')$  gauge dynamics. This dynamics can be captured from a weakly coupled holographic dual description by replacing the D4'-branes by the near horizon geometry and still keeping the D4-branes as probes in order to keep the  $SU(N_c)$  in a weak coupling regime. The geometry is given by Witten [15] and the throat is developed at the location of the D4'-branes to connect the D8- and  $\overline{\text{D8}}$ -branes smoothly which describes the spontaneous chiral symmetry breaking.

In this geometry, the worldsheet instantons which we like to evaluate are the ones whose boundary is on the D4- and the D8-branes in Fig.6 with two twist vertex operators which describe the quarks. By evaluating the amplitude in a saddle point approximation, we can show that the quarks become massive since the amplitude is given as

$$S_{\text{instanton}} = c \int d^4x \text{tr } \bar{q}_L q_R \exp[-S_{NG}] + \text{h.c.} \quad (2.1)$$

Here the trace is taken over the flavor  $U(N_f)$  indices and  $S_{NG}$  is the classical value of the Nambu-Goto action. The Hermitian conjugate comes from the oppositely oriented world-sheets, and  $c$  is a constant factor. We have to evaluate  $S_{NG}$  in the curved geometry of [15] which is

$$ds^2 = \left( \frac{U}{R'} \right)^{3/2} (dx_4^2 + f(U)d\tau^2) + \left( \frac{R'}{U} \right)^{3/2} \left( \frac{dU^2}{f(U)} + U^2 d\Omega_4^2 \right), \quad (2.2)$$

$$e^\phi = g_s \left( \frac{U}{R'} \right)^{3/4}, \quad F_4 = \frac{2\pi N'}{V_4} \epsilon_4, \quad f(U) = 1 - \frac{U'^3_{KK}}{U^3}, \quad (2.3)$$

$$R'^3 = \pi g_s N' l_s^3, \quad M_{KK} = \frac{3U'^{\frac{1}{2}}_{KK}}{2R'^{\frac{3}{2}}}, \quad (2.4)$$

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<sup>||</sup>When we have  $N_f > 1$  number of D8- and  $\overline{\text{D8}}$ -branes, we have other worldsheets ending on, for example, D4<sup>(i)</sup>-D8-D4'<sup>(j)</sup>-D8 branes. These worldsheets do not give rise to the quark masses (and the pion masses).



where  $U \geq U'_{KK}$  is the radial direction transverse to the D4'-branes,  $g_s$  is the string coupling,  $l_s = \sqrt{\alpha'}$  is the string length,  $V_4$  and  $\epsilon_4$  are the volume and line element of  $S_4$ .

Let the probe  $N_c$  D4-brane be placed at  $U = U_0$ . Since the minimal size worldsheet instanton extends along  $U$  and  $\tau$  directions, we can evaluate

$$S_{NG} = \frac{1}{2\pi\alpha'} \int_0^{\delta\tau/2} d\tau \int_{U'_{KK}}^{U_0} dU \sqrt{g_{\tau\tau}g_{UU}} = \frac{U_0 - U'_{KK}}{2\alpha' M_{KK}} = \frac{\pi M_W}{M_{KK}} \quad (2.5)$$

in the approximation  $U_0 \gg U'_{KK}$ , where  $M_W$  is the mass of the massive gauge boson  $W_\mu$  computed as  $M_W = U_0/(2\pi\alpha')$ . We finally obtain

$$m_q = c \exp \left[ -\frac{\pi M_W}{M_{KK}} \right]. \quad (2.6)$$

The exponential suppression factor is the same as that in the field theory calculation which supports our identification of the worldsheet instantons with the four-fermi Feynman graphs. The constant factor  $c$  depends on the string length, the string coupling and the chiral condensate  $\langle \bar{Q}_L Q_R \rangle$ , and then is an important coefficient. However it is difficult to compute  $c$  in the curved geometry and this leads us to the study of introducing D6-branes instead of the D4'-branes in the next section.

## 2.3 Pion mass

We found that the quark becomes massive due to the chiral symmetry breaking caused by the  $N'$  D4-branes. Thus we should be able to see that the pions become massive and the mass should satisfy the GOR relation.

In the low energy limit, both  $SU(N_c)$  and  $SU(N')$  become strongly coupled and the holographic dual description gives a weakly coupled description in terms of gauge invariant operators, *i.e.* mesons. In doing that, we need a multi-center solution of the D4-branes and its near horizon geometry. However the explicit metric is not known and in the next section we study a more computable model. Thus in this subsection we only give a qualitative discussion on how the pions are shown to be massive in the holographic dual description.

The geometry of the two-center solution of D4-branes has two throats at the location of the D4- and the D4'-branes respectively. See Fig.7. The geometry is roughly obtained by gluing two metrics (2.2). Thus we again have the worldsheet instantons denoted by the shaded region in the same figure. The amplitude of this worldsheet instantons is given by

$$S_{\text{instanton}} \propto \int d^4x \frac{1}{g_s} \text{Ptr} \exp \left[ -S_{NG} + i \oint dz A_z \right] + \text{h.c.}, \quad (2.7)$$

where we have included a boundary coupling to  $A_z$  and  $z$  parameterizes the boundary of the worldsheet instanton. The factor  $1/g_s$  is introduced because the worldsheet is a disk. Since the pion wave function is localized at the D4 throat [2], we have

$$\text{Ptr} \exp \left[ i \oint dz A_z \right] = \text{tr} U, \quad U \equiv \exp [i2\pi(x)/f_\pi], \quad (2.8)$$

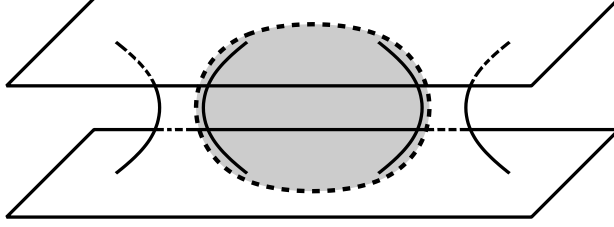


Figure 7: The shaded region denotes a worldsheet instanton. The two throats are developed at the location of D4 and D4'-branes.

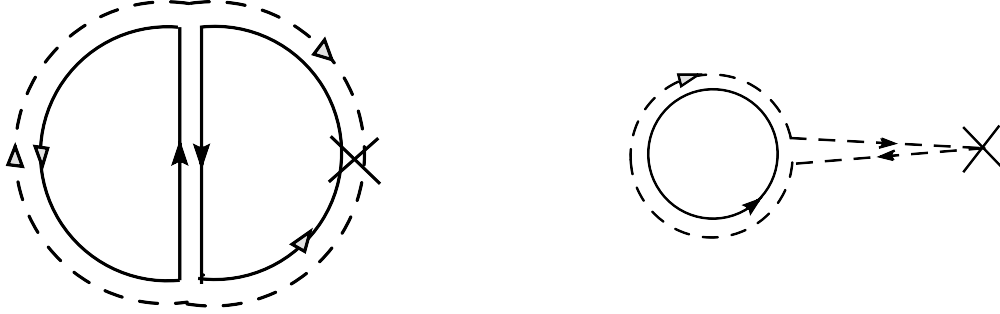


Figure 8: The Feynman diagrams responsible for quark masses can be written in the double-line notation, which shows that the flavor line (dashed lines) form a closed single loop. Solid lines are for the color indices.

after substituting into  $A_z$  the pion wave function. Then we obtain

$$S_{\text{instanton}} = m_\pi^2 f_\pi^2 \text{tr}(U + U^\dagger), \quad m_\pi^2 \propto \frac{1}{g_s} \exp[-S_{NG}] \propto m_q \quad (2.9)$$

because the exponent  $\exp[-S_{NG}]$  is roughly same as that of the quark mass. We can see the GOR relation is naturally satisfied. This is the lowest mass deformations in the pion chiral Lagrangian which should appear by the quark mass perturbations to QCD.

Before closing this section, we give some interesting observations. The holographic description is reliable in the large  $N_c$  (and  $N'$ ) and large 't Hooft coupling, and a genus zero string worldsheet would correspond to a planar diagram in a large  $N$  QCD proposed by 't Hooft. The Feynman graphs using the double-line notation in a large  $N$  QCD is shown in Fig. 8. It is clear from this that these Feynman diagrams are included in a planar diagram with one boundary. On the other hand, the worldsheet instanton is a disk amplitude which is nothing but a planar diagram with one boundary. This gives a consistency that we identify the four-fermi Feynman graphs with the worldsheet instantons.

The worldsheet instanton does not contribute to the Dirac-Born-Infeld (DBI) action  $S_{\text{DBI}}$  for the D8-branes, *i.e.* the total action  $S_{\text{D8}}$  is

$$S_{\text{D8}} = S_{\text{DBI}} + S_{\text{instanton}}. \quad (2.10)$$

There must not be a mass term for the pions from  $S_{\text{DBI}}$  at the leading order in large  $N$ . We can easily see this from the fact that the pion comes from a KK zero mode in the gauge field on D8-branes along the extra dimension. Since the gauge fields always appear through field strengths in the DBI action, the pion  $\pi(x)$  always appears with four-dimensional derivatives, i.e.  $\partial_\mu \pi(x)$ , and cannot have a mass term.

In this section, we focused on the quark  $(q_L, q_R)$  and the pion in the  $SU(N_c)$  sector. Because the D4'-branes realize another  $SU(N')$  gauge theory, we have  $(Q_L, Q_R)$  and another pion in  $SU(N')$  sector. Since the chiral symmetry is still spontaneously broken even though we introduced the D4'-branes, one combination of the two pions are still massless Nambu-Goldstone bosons. In order to decouple these modes from the QCD sector, we need to take a limit which realizes the pion decay constant for this unwanted pion infinite by taking  $N'$  to the infinity as well as taking the distance from D4-branes infinite while keeping the pion mass we computed finite. Since we can not explicitly compute  $S_{\text{instanton}}$ , the actual realization of this limit is not clear.

Therefore as we will explain in the next section, we consider another deformation by instead introducing D6-branes which keeps the idea in this section while needs no limit, *i.e.* the chiral symmetry is explicitly broken by the D6-branes.

### 3 Quark mass deformation by D6-branes and Pion mass

In this section, we propose another way to introduce quark masses to the Sakai-Sugimoto model. In the approach of this section, the evaluation of the quark and pion mass terms is more tractable. First we explain the idea of introducing a probe D6-brane ending on the D8- and the  $\overline{\text{D8}}$ -branes, instead of the D4'-branes. Then, we compute the worldsheet instanton explicitly to obtain the quark masses, pion mass and the chiral condensate. Finally, we present a numerical evaluation of those values, for a tentative comparison with experimental or lattice QCD's results.

#### 3.1 Configuration of the D6 ending on D8 and $\overline{\text{D8}}$

As seen in the previous section we can introduce the quark mass term if we have a 1-cycle in the brane configuration, because the worldsheet instantons are possible. From this point of view, we do not necessarily use the D4'-branes and instead introduce  $N'$  D6-branes which are separated away from the D4-branes and end on the D8-branes and the  $\overline{\text{D8}}$ -branes (each D6-brane can end on different D8- and  $\overline{\text{D8}}$ -branes). The orientation of the additional probe D6-brane is shown in the table.\*

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\* To have a stable configuration of the D6-brane, the D6-brane is curved when the D4-brane is replaced by its geometry. In the appendix B, we find a consistent curved shape of the D6-brane in the background, by solving the equations of motion of the D6-brane effective action.

	0	1	2	3	4	5	6	7	8	9
$N_c$ D4	○	○	○	○	○					
$N_f$ D8 $\overline{\text{D8}}$	○	○	○	○		○	○	○	○	○
$N_f$ D6	○	○	○	○	○		○	○		

This brane-ending-on-brane configuration is represented by a stable spike solution on the D8-branes. This means that the D8-branes and the  $\overline{\text{D8}}$ -branes are smoothly connected with each other by the smooth spike (volcano-shaped) [16], and thus there is no twist operator insertion at the end points of the D6. We can obtain a worldsheet instanton whose shape is almost the same as that of the case introducing the D4'-branes.<sup>†</sup>

Let us summarize our field theoretical understanding on the D-branes with emphasis on the differences from the D4'-brane case.

- The D6-brane can be treated as a probe ( $N' \ll N_c$ ), hence it does not modify the background geometry. The D6-brane is a classical solution on the D8-branes (or the  $\overline{\text{D8}}$ -branes), and it is a part of the probe flavor D-branes. Thus, on the contrary to the previous section, the background is still given by the  $N_c$  D4-branes.
- The introduction of the D6-brane breaks the chiral symmetry, since the D6-branes end on the D8 and  $\overline{\text{D8}}$ -branes and thus connect the D8 and  $\overline{\text{D8}}$ -branes. The breaking is explicit, not spontaneous, since the D6-branes extend along  $(x^6, x^7)$  infinitely and the fields localized on the D6-branes, which would be Nambu-Goldstone bosons, do not have a finite kinetic term in the four dimensions.
- From the theory on the D8-branes, the D6-branes are interpreted as monopoles with the charge  $(1, -1)$  under the chiral symmetry for a single flavor case  $U(1)_L \times U(1)_R$ . Thus the source for the Dirac monopole explicitly breaks the axial symmetry. For D6-branes not on top of each other, all the axial symmetry is broken, which corresponds to the situation where quark masses are not coincident with each other.
- The field theoretical understanding of how the effects of chiral symmetry breaking on the D6-brane sector are mediated into the QCD sector on the D4-branes is essentially the same. Here we have massive scalar fields (instead of massive gauge bosons) from the open strings between the D4-branes and the D6-branes which, for example, play the role of the mediators.
- We have an easy way to realize flavor-dependent quark masses by shifting the location of each D6-brane independently. Another way (which can also apply for the D4'-branes) is to shift the D8-brane from the anti-podal points, but the D8-brane configuration away from the anti-podal point is only numerically known [18] which makes the estimate more difficult.

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<sup>†</sup> This instanton is similar to the one considered in [17] where QCD instanton is studied. We thank J. Sonnenschein for pointing this out.

### 3.2 Quark mass from worldsheet instanton

Let us first compute the quark mass term. At this stage we keep the D4-branes to be probes, that is, the spacetime is still flat. (In the next subsection we replace them with a curved geometry.) The mass generation mechanism by the worldsheet instantons is quite the same as in the previous section, see Fig. 6. The throat on the right hand side of the figure is now  $N'$  spiky D6-branes ending on the D8-branes and the  $\overline{\text{D8}}$ -branes. At the end points of the D6-brane, there is no insertion of a vertex operator because the D6 and the D8 join smoothly there. Resultantly, the worldsheet instanton amplitude should be given by

$$\frac{\mathcal{N}N'}{g_s l_s} e^{-S_{\text{NG}}} \int d^4x \bar{q}_L q_R. \quad (3.1)$$

The classical worldsheet action  $S_{\text{NG}}$  is just the area of the worldsheet times string tension in flat space, so we have  $S_{\text{NG}} = (1/(2\pi\alpha'))U_0(\pi/M_{\text{KK}}) = \pi M_W/M_{\text{KK}}$  where  $U_0$  is the length of an open string stretching between the D4- and D6-branes and then  $M_W$  is the mass of modes from the open string. The factor  $N'$  comes since the boundary of worldsheet instantons can be on  $N'$  different D6-branes. The front factor  $1/g_s$  is for a disk amplitude, and the  $1/l_s$  factor should be provided from the normalization of the quark vertex operator, to make a sensible dimensionality of the amplitude. The dimensionless number  $\mathcal{N}$  is a normalization factor from the quark vertex operators, the quark kinetic terms and the fluctuations around our semi-classical worldsheet.

For the multi-flavor case, by shifting the position of each D6-brane respectively, we can introduce quark masses depending on flavors. For each flavor labeled by  $i$ , we can associate the location  $U = U_0^{(i)}$  of coincident  $N'$  D6-branes (then the total number of the D6-branes is  $N'N_f$ ). Then obviously, the quark mass term is a sum of each instanton,  $\sum_i \int d^4x \bar{q}_L^{(i)} (m_q)_{ij} q_R^{(j)}$ , with the following diagonal quark mass matrix

$$(m_q)_{ij} = \mathcal{N}N' \delta_{ij} \frac{2\pi M_{\text{KK}}}{g_{\text{YM}}^2} \exp \left[ -\frac{\pi M_W^{(i)}}{M_{\text{KK}}} \right], \quad (3.2)$$

where  $M_W^{(i)} = U_0^{(i)}/(2\pi\alpha')$ .

### 3.3 Pion mass, GOR relation and chiral condensate

Next, we study the low energy limit in the holographic dual by replacing the D4-branes by their geometry (Witten's geometry) and compute the pion mass term. Since there is a nontrivial 1-cycle on the D8-brane, the worldsheet instanton gives us a pion mass term (see Fig. 7). The mechanism is completely analogous to the previous section. In the following, we present explicit evaluation of the worldsheet instanton amplitude. We will adopt some crude approximations for obtaining the results because a curved background makes the explicit evaluation quite difficult.

The worldsheet instanton amplitude is given as

$$S_{\text{instanton}} = N' \frac{1}{g'_s} \frac{1}{(2\pi)^3 l_s^4} \int \sqrt{-\det g} d^4x e^{-S_{\text{NG}}} \left[ \text{Ptr} \exp \left[ -i \oint A_z dz \right] + \text{c.c.} \right], \quad (3.3)$$

where the trace is taken over the flavor indices. To obtain this expression, we made the following approximations as we will explain step by step.

The classical worldsheet action  $S_{\text{NG}}$  is evaluated in the same way computed in the previous section, *i.e.*, see (2.5), so  $S_{\text{NG}} = \pi M_W / M_{\text{KK}}$ .

Let us explain the front factor  $1/(g'_s(2\pi)^3 l_s^4)$  in (3.3). First, we notice that the worldsheet instanton is wrapping the minimal cycle whose zero mode is 4-dimensional. The worldsheet instanton cannot move along the direction of the  $S^4$ : the boundary should be on the D6-brane whose closest point to the D4-branes is just a point on the  $S^4$  (see appendix B for details). So the instanton amplitude should be proportional to a D3-brane tension  $\mathcal{T}_{\text{D3}}$ , not  $\mathcal{T}_{\text{D8}}$ .<sup>‡</sup> The D3-brane tension is given by the front factor  $1/(g'_s(2\pi)^3 l_s^4)$ .

Note that the string coupling constant should be the effective coupling constant  $g'_s$  obtained by including the effect of the background dilaton,  $1/g'_s = (1/g_s) (R/U_{\text{KK}})^{3/4}$  (see the classical solution (2.2) but remove the prime in the corresponding quantities.). In getting this expression we made a crude (but reasonable) approximation that the  $U$ -dependence of the dilaton can be approximated by the value at  $U \sim U_{\text{KK}}$ . This is because the pion wavefunction is localized around  $U_{\text{KK}}$  so that most of the contribution for the integral over  $U$  would come from the region  $U \sim U_{\text{KK}}$ , even though the worldsheet itself is elongated in the region  $U_{\text{KK}} < U < U_0$ . Furthermore, the invariant volume  $\sqrt{\det g}$  in the  $(x^0, x^1, x^2, x^3)$  directions was inserted in (3.3) for consistency. Using the metric expression (2.2), the four-volume is given as  $\sqrt{-\det g} = (U_{\text{KK}}/R)^3$ .

We can calculate the instanton amplitude (3.3) using the relations between the gravity and the gauge theory quantities, namely,

$$R^3 = \frac{g_{\text{YM}}^2 N_c l_s^2}{2M_{\text{KK}}}, \quad U_{\text{KK}} = \frac{2}{9} g_{\text{YM}}^2 N_c M_{\text{KK}} l_s^2, \quad g_s = \frac{g_{\text{YM}}^2}{2\pi M_{\text{KK}} l_s}, \quad (3.4)$$

then we arrive at

$$S_{\text{instanton}} = \frac{2N'}{3^{9/2} \pi^2} g_{\text{YM}} N_c^{3/2} M_{\text{KK}}^4 \exp \left[ -\frac{\pi M_W}{M_{\text{KK}}} \right] \int d^4x (\text{tr} U + \text{tr} U^\dagger). \quad (3.5)$$

This is the pion mass term generated by the worldsheet instanton in the curved background. Again, we found an expression well-known in chiral perturbation theory. Substituting the

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<sup>‡</sup> Along the  $S^4$  directions transverse to the worldsheet instanton configuration, the string fluctuation follows Dirichlet boundary condition rather than Neumann. This effectively reduces the zero mode integral of the vertex insertion at the boundary of the worldsheet instanton. The number of the transverse directions is 4 in the worldvolume of the D8-branes. Taking into account the  $z$  direction along which the worldsheet boundary is elongated, we need to introduce the factor  $\mathcal{T}_{\text{D3}}$  in front of the worldsheet instanton amplitude, rather than just  $1/g_s$ .

quark mass formula (3.2), this can be re-written as

$$S_{\text{instanton}} = \frac{1}{3^{9/2}\pi^3} g_{\text{YM}}^3 N_c^{3/2} M_{\text{KK}}^3 \mathcal{N}^{-1} m_q \int d^4x (\text{tr} U + \text{tr} U^\dagger). \quad (3.6)$$

Expanding  $U = \exp[2i\pi(x)/f_\pi]$  to the second order in the pion field  $\pi(x)$ , we obtain the GOR relation

$$m_\pi^2 = \frac{4g_{\text{YM}}^3 N_c^{3/2} M_{\text{KK}}^3}{3^{9/2}\pi^3 \mathcal{N}} \frac{1}{f_\pi^2} m_q. \quad (3.7)$$

The chiral condensate computed in our model is, therefore,

$$\langle \bar{q}q \rangle = \frac{2}{3^{9/2}\pi^3} \mathcal{N}^{-1} g_{\text{YM}}^3 N_c^{3/2} M_{\text{KK}}^3. \quad (3.8)$$

This is the expression for the chiral condensate in the Sakai-Sugimoto model. <sup>§</sup>

Note that, as we stated, we made crude approximations in evaluating the coordinate dependence of the background dilaton and metric. But it is nontrivial that all the  $l_s$  dependence disappears at the end, which may signal a consistency of the approximation.

### 3.4 Flavor-dependent quark masses and numerical results

In this subsection we consider the flavor dependent quark masses by placing the D6-branes at different points. The worldsheet instanton ends in part on the D6-brane, and let us consider the transverse scalar field  $\Phi$  on the D6-branes. The separation of the D6-branes can be encoded in the worldsheet boundary interaction as a condensation of the transverse scalar field  $\Phi$ . Because the D6-branes are made of the spike of the D8- and  $\overline{\text{D8}}$ -branes, this  $\Phi$  can be at the same time regarded as a transverse scalar field on the D8- $\overline{\text{D8}}$ -branes, too. So, in total, together with the usual boundary interaction for the gauge fields on the D8-branes, the boundary interaction is

$$\text{Ptr exp} \left[ -i \oint A_z dz - \int_b \Phi dz \right] \quad (3.9)$$

The integration region  $b$  is a period (in the worldsheet boundary) where the worldsheet ends on the D6-branes.\* We can parametrize the scalar field as  $2\pi\alpha'\Phi = \text{diag}(U_0^{(1)}, U_0^{(2)}, \dots)$  where  $U_0^{(i)}$  denotes the location of the  $N'$  D6-branes<sup>†</sup> for the  $i$ -th flavor. If we neglect the dynamical fluctuation of the scalar field, then this is just a constant matrix.

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<sup>§</sup> An interesting feature of this chiral condensate is that, if  $\mathcal{N} = 1$ , it is independent of  $N_c$  if it is written with 'tHooft coupling  $\lambda = g_{\text{YM}}^2 N_c$ . It sounds consistent with Coleman-Witten argument [19]. The  $N_c$  dependence of the chiral condensate found here is a little different from what was found in [13].

\* We don't have the factor  $i$  in front of the scalar field coupling, because the scalar field is associated with the instanton which is Euclideanized, and we will see the consistency below.

<sup>†</sup>This is the location of the tip of the smeared D6-brane cone, see appendix B for the shape of the D6-brane. The tip is given by  $\theta_1 = 0$ .

The vertex insertion of the gauge field  $A_z dz$  on the boundary of the worldsheet instanton is located almost around the tip of the cigar (due to the localized distribution of the pion wave function found in [2]), that is far away from the region where the transverse scalar  $\Phi$  is inserted (where the D6-brane is present). Therefore, the path-ordering in (3.9) is effectively approximated by the following ordering

$$\text{tr} \left[ \text{P exp} \left[ -\Phi \int_b dz \right] \cdot \text{P exp} \left[ -i \oint A_z dz \right] \right]. \quad (3.10)$$

Note that the integral  $\int_b$  is performed only along the worldsheet boundary on the D6-branes. This means

$$\exp \left[ -\Phi \int_b dz \right] = \exp \left[ -\Phi \frac{\pi}{M_{\text{KK}}} \right] = \exp \left[ -\text{diag} \left( \frac{\pi M_W^{(1)}}{M_{\text{KK}}}, \frac{\pi M_W^{(2)}}{M_{\text{KK}}}, \dots \right) \right] = \frac{g_{\text{YM}}^2}{2\pi M_{\text{KK}}} m_q \quad (3.11)$$

Note that we realize a non-abelian generalization of the worldsheet area  $S_{\text{NG}} \simeq \pi M_W / M_{\text{KK}}$  in this manner.  $S_{\text{NG}} = 0$  in this notation. If instead one uses the minimal value of the  $S_{\text{NG}}$  which is  $\pi M_W^{(1)} / M_{\text{KK}}$ , then one should parameterize the transverse scalar field as its deviation from  $U^{(1)}$ , as  $2\pi\alpha'\Phi = \text{diag}(0, U_0^{(2)} - U_0^{(1)}, U_0^{(3)} - U_0^{(1)}, \dots)$  to get the correct non-Abelian expression for the worldsheet boundary interaction. Using this expression, we obtain the following term induced by the worldsheet instanton:

$$S_{\text{instanton}} = \frac{1}{3^{9/2}\pi^3} g_{\text{YM}}^3 N_c^{3/2} M_{\text{KK}}^3 \mathcal{N}^{-1} \int d^4x \text{tr} [m_q(U + U^\dagger)]. \quad (3.12)$$

This form,  $\text{tr}[m_q(U + U^\dagger)]$ , is in fact what is usually expected in chiral perturbation theory.

In the two-flavor case, we have  $m_q = \text{diag}(m_u, m_d)$ , so

$$\text{tr} [m_q(U + U^\dagger)] = 2(m_u + m_d) \left( 1 - \frac{1}{f_\pi^2} \pi_a(x)^2 + \mathcal{O}(\pi(x)^4) \right). \quad (3.13)$$

where the component definition is  $\pi(x) = \pi_a(x)\sigma_a/2$ . So at this leading order estimate the mass of  $\pi^0$  is equal to the mass of  $\pi^\pm$ , as in the usual chiral perturbation theory. Using the expression [2] for the pion decay constant  $f_\pi = g_{\text{YM}} N_c M_{\text{KK}} / (3\sqrt{6}\pi^2)$ , (3.12) gives a pion mass term

$$m_\pi^2 = 2(m_u + m_d) \frac{2\pi g_{\text{YM}} M_{\text{KK}}}{3\sqrt{3} N_c^{1/2} \mathcal{N}}. \quad (3.14)$$

Just for an illustration, we present a numerical evaluation of the relation (3.7).<sup>‡</sup> Sakai and Sugimoto [2] deduced numerical values as  $g_{\text{YM}} = 2.35$ ,  $M_{\text{KK}} = 949$  [MeV]. The first came from

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<sup>‡</sup> Readers are advised that the values presented below are just for illustration, and should not be taken seriously, as we made a crude approximation in evaluating the worldsheet instanton in the curved background, and we are working in the leading order in large  $N$  and large 'tHooft coupling expansion.



the equation  $\kappa = \lambda N_c / (216\pi^3) = 0.00745$  with  $N_c = 3$ . Then, with  $m_\pi = 140$  [MeV] as an input, the mass formula (3.14), with  $\mathcal{N} = 1$  as an assumption, gives

$$m_u + m_d = 6.29 \text{ [MeV]}. \quad (3.15)$$

Experimental results shown in the particle data book are  $m_u = 1.5 - 4.0$  [MeV],  $m_d = 4 - 8$  [MeV]. Surprisingly, our result is very close to the observed value of the up/down quark mass.

The difference between  $\pi_0$  and  $\pi_\pm$  can be seen in higher-order chiral Lagrangian in chiral perturbation theory. In appendix C, we give a computation of two worldsheet instantons, to get the higher-order terms.

The chiral condensate (3.8) is numerically evaluated as

$$\langle \bar{q}q \rangle = (299 \text{ [MeV]})^3, \quad (3.16)$$

which is in quite good agreement with values obtained in quenched/unquenched lattice simulation,  $\langle \bar{q}q \rangle \simeq (2.5 \times 10^2 \text{ [MeV]})^3$ .

## 4 Application to the holographic QCD of flavor D6-branes

Our idea is based on a field theoretical picture inspired by extended technicolor theories. The technicolor sector is coupled with the QCD sector via massive gauge-bosons of the extended gauge group, and the condensation of the techni-quarks  $Q$  gives rise to a quark mass term through the induced four-Fermi coupling. Therefore, this mechanism can be applied to other holographic models of QCD. In this section we demonstrate this by applying our idea to the D4-D6 model of Kruczenski *et.al* [13]. The model consists of  $N_c$  D4-branes giving rise to the curved geometry and  $N_f$  flavor D6-branes which are introduced as probes. In fact the model can describe massive quarks, as the D6-branes can be shifted away from the D4-branes. We will see that our technicolor D4'-branes can shift the D6-branes further and the quarks get extra masses because of this.

The D-brane configuration of the model in [13] consists of the D4 and D6-branes as in the table:

	0	1	2	3	4	5	6	7	8	9
$N_c$ D4	○	○	○	○	○					
$N_f$ D6	○	○	○	○		○	○	○		

The radial directions in  $(x^5, x^6, x^7)$  and in  $(x^8, x^9)$  from the D4-branes are denoted as  $\lambda$  and  $r$  respectively, and the D6-brane configuration in the D4-brane geometry is solved in their paper. The asymptotic behavior for  $\lambda \gg \lambda_c$ , ( $\lambda_c$  is a some large value), is

$$r \sim r_\infty + \frac{c}{\lambda}, \quad (4.1)$$

where  $r_\infty$  is the asymptotic distance between the D4- and the D6-branes which is related to the bare quark mass  $m_q$  as

$$m_q = \frac{r_\infty}{2\pi l_s^2}, \quad (4.2)$$

and  $c$  is related to the value of condensate<sup>§</sup>

$$\langle \bar{q}q \rangle \simeq \frac{g_{YM}^2 N_c^2 M_{KK}^3}{U_{KK}^2} c \quad \left( U_{KK} = \frac{2}{9} g_{YM}^2 N_c M_{KK} l_s^2 \right) \quad (4.3)$$

The mass of the pion, which is a pseudo Nambu-Goldstone boson associated with the breaking of the rotation symmetry of the  $(x^8, x^9)$ -plane, is also computed in their paper

$$m_\pi^2 \sim \frac{m_q M_{KK}}{g_{YM}^2 N_c}. \quad (4.4)$$

We now introduce our technicolor  $N'$  D4'-branes which are parallel to, but separated from the  $N_c$  D4-branes along  $\lambda$  with  $r = 0$  (so that  $U = \lambda$ ). Let's say that the D4-branes are sitting at  $\lambda = 0$  and the D4'-branes are at  $\lambda = \lambda_0$ . We first treat the D4-branes as probes and consider the effects from confining D4' dynamics. Assuming the distance  $U_0 = \lambda_0$  between the D4 and D4'-branes is much larger than  $\lambda_c$ , i.e.  $U_0 \gg \lambda_c$ , the value of  $r$  at the position of the D4-brane  $\lambda = 0$  *induced by the presence of D4'-brane* would be

$$r \sim r_\infty + \frac{c'}{U_0}, \quad (4.5)$$

where  $c'$  denotes the contribution from the D4'-branes, *i.e.*

$$\langle \bar{Q}Q \rangle \simeq \frac{g_{YM}^2 N'^2 M_{KK}^3}{U_{KK}'^2} c' \quad \left( U_{KK}' = \frac{2}{9} g_{YM}^2 N' M_{KK} l_s^2 \right) \quad (4.6)$$

For the probe D4-branes before their gravity background is considered, this would serve as its new asymptotic distance  $r_\infty$  from the D6 branes. It is shifted from the original  $r_\infty$  by  $\frac{c'}{U_0}$ . We then take the gravity background for our D4-branes, and we have

$$r \sim r_\infty + \frac{c'}{U_0} + \frac{c}{\lambda}, \quad (4.7)$$

for  $\lambda_c \ll \lambda \ll U_0$ .

Therefore we see that the quark mass in QCD sector on the D4-branes is shifted as

$$m_q \sim \frac{U_{KK}}{2\pi l_s^2} \left( r_\infty + \frac{c'}{U_0} \right) = \frac{U_{KK} r_\infty}{2\pi l_s^2} + \frac{g_{YM}^2 \langle \bar{Q}Q \rangle}{81\pi^2 M_{KK} M_W} \quad (4.8)$$

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<sup>§</sup>  $U_{KK} = 1$  is used in [13].

where we have used the W-boson mass  $M_W = U_0/(2\pi l_s^2)$  of the D4-D4' string. The mass of the pion in the QCD sector is read again

$$m_\pi^2 \sim \frac{m_q M_{KK}}{g_{YM}^2 N_c}. \quad (4.9)$$

In the above computation, we used the holographic dual description and used the fact that the pseudo Nambu-Goldstone wavefunction in QCD sector is localized at  $\lambda = 0$  region. We can also study the quark mass from a field theoretical point of view, since introducing the D4'-branes has an interpretation of introducing a techni gauge theory. The invariance under the rotation in the  $(x^8, x^9)$ -plane is broken by the D4'-branes and the breaking effects are communicated into the QCD sector on the D4-branes by, for example, massive gauge bosons which come from the open strings connecting the D4- and D4'-branes. We then have a Feynman graph similar to Fig.2 and can compute the quark mass after replacing  $\bar{Q}Q$  with its condensate  $\langle \bar{Q}Q \rangle$ . After a careful calculation, the quark mass is computed as a consistent value with (4.8) up to a numerical factor.

A comment is in order. The D6-brane configuration is obtained by solving the equation of motion computed from the DBI action for the D6-brane in the D4-brane geometry. This means that the pion mass terms are induced in the DBI action because of the D4'-branes, on the contrary to the Sakai-Sugimoto model where the DBI action does not have a mass term for the pion. This is understood from the string world sheets. The string world sheet which corresponds to the graph in Fig.2 is not a worldsheet instanton, but just a disk amplitude whose boundary is on the D6-branes. The minimal worldsheet area is vanishing. If the boundary passes through the two throats created by the D4 and D4'-branes, this disk worldsheet picks up the effects on the D4'-branes and communicate them into the D4-branes. Since the DBI action is computed from this infinitely small disk worldsheet (which is not a worldsheet instanton), the mass terms for the pion appear in the DBI action in the present case.

## 5 Conclusions and Discussions

In this paper, we propose the deformations of the Sakai-Sugimoto model to generate the quark masses. Our considerations are motivated by extended technicolor theories where we break chiral symmetry via introducing a technicolor sector. We systematically trace the mass deformations in different descriptions, that is, weak coupling D-branes setting and the corresponding holographic gravity description. In the field theory side, the chiral condensate in the technicolor sector is mediated to the QCD sector, generating the quark masses. One then expects the massive quarks will yield the massive pion. Moreover, the pion and the quark masses are expected to obey the GOR relation from the chiral Lagrangian consideration. To *derive* the GOR relation from first principles, we should have a good control on the strong coupling dynamics of QCD. Relying on holographic principle, our analysis verifies the GOR relation impressively. Furthermore, we find a good numerical agreement of the chiral condensate with the experimental or lattice results.

In our construction, by introducing additional technicolor branes, a novel mechanism of worldsheet instanton gives us a controlled contribution to the quark mass deformation. In the strong coupling regime, the  $N_c$  color branes are replaced by a curved geometry, and the corresponding worldsheet instanton is now dressed by the Wilson line of the probe D8-brane gauge field via worldsheet boundary interactions. This results in an additional term in the D8-brane effective action corresponding to the mass deformation of the chiral Lagrangian of the pions.

We realize our idea with different types of D-branes, the  $D4'$ - and  $D6$ -branes. The former has a clear field theory interpretation in terms of a GUT-like extended technicolor theory, while the latter case is more tractable in actual calculations. Moreover, it allows us flavor-dependent quark mass terms. QCD  $\theta$  angle can also be introduced easily by turning on  $C_{RR}^{(2)}$  in the background. Based on these brane settings, we have verified that the deformations indeed correspond to the lowest mass perturbation in the chiral Lagrangian of pions via a novel mechanism of worldsheet instantons. We also have the GOR relation satisfied, and from this relation we have extracted the quark masses and the chiral condensate of the Sakai-Sugimoto model for the first time, which happens to be surprisingly close to the lattice QCD estimate.

We expect that the progress made in this paper can push the Sakai-Sugimoto model towards a more realistic nonperturbative description of QCD. We hope it may inspire more comparisons with the experimental or lattice QCD results. For example, we may expect the worldsheet instanton will also affect other hadron spectrum and their dynamics simply because both mesons and baryons are excitations of the quarks. As shown in [3], there is a difficulty in fitting both the baryon and meson spectrum well at the same time in Sakai-Sugimoto model. It is then interesting to see if our mass deformation would help to resolve the issue.

There are many issues in lattice QCD or nuclear physics which call for a reliable and calculable QCD model with non-zero quark masses. For example, the mass shift of the pions in a finite temperature or in quark-gluon plasma phase is of much interest. These can be done only with the models with non-zero quark mass as the one we proposed. The study of renormalization of the value of chiral condensate is also interesting. Also, lattice computations have been done in quench approximation where quarks are heavy. It is often stated that the Sakai-Sugimoto model corresponds to a quenched approximation. However, it is different from the lattice quenching of heavy quarks, because in the original Sakai-Sugimoto model quarks are massless. It is also hoped that our mass deformation will help the holographic QCD to address some issues like nuclear potential in nuclear physics, where the pion mass will play an important role in their dynamics.

*Note added in proof:* While we are preparing our manuscript, there appeared a paper [20] which has some overlap with ours in discussing the mass deformation by the worldsheet instantons.

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## A Field-theoretical computation of the four-fermi term

The theory living on the D4-branes and the D4'-branes is 5 dimensional. The four-Fermi coupling appears effectively after one integrates out the massive W-bosons. The coupling between the W-boson and quarks (techni-quarks) is given by gauge couplings

$$\frac{g}{2} \int d\tau d^4x [\delta(\tau) \bar{q}_L \gamma_\mu W^\mu Q_L + \delta(\tau - \pi/M_{KK}) \bar{q}_R \gamma_\mu W^\mu Q_R]. \quad (\text{A.1})$$

Note the delta-functions are inserted in the coupling. The  $N_f$  D8-branes are located at  $\tau = 0$  while the  $N_f$   $\overline{\text{D8}}$ -branes are located at the anti-podal point,  $\tau = \pi/M_{KK}$ . On the D8-D4 intersection, the quark  $q_L$  lives, while on the D8-D4' intersection, the quark  $Q_L$  lives. As for the intersection with the  $\overline{\text{D8}}$ -branes,  $q_R$  and  $Q_R$  live in the same manner. This zero-mode condition for the quarks is represented as the delta-functions above.

The coupling (A.1) is gauge invariant and also invariant under the chiral flavor symmetry  $U(N_f)_L \times U(N_f)_R$ . The coefficient  $g$  is the gauge coupling of the 5-dimensional Yang-Mills theory on the D4-branes and the D4'-branes, and so it is related to the 4-dimensional Yang-Mills coupling constant by a dimensional reduction,  $(2\pi/M_{KK})(1/g^2) = 1/g_{\text{YM}}^2$ , that is to say, the KK zero mode along the direction  $\tau$  is the gluon in 4-dimensions.

To derive the effective four-Fermi coupling, we decompose the 5-dimensional notation (A.1) into 4-dimensional fields, following the standard prescription given by [14]. We just employ a KK expansion of all the fields and compute the expanded theory as if it were a theory of infinite number of 4-dimensional fields. The KK expansion for the W-boson is given with the periodic boundary condition as<sup>¶</sup>

$$W_\mu(x, \tau) = \sum_{n=0}^{\infty} \left[ \sin(n\tau M_{KK}) W_\mu^{(n)}(x) + \cos(n\tau M_{KK}) \tilde{W}_\mu^{(n)}(x) \right]. \quad (\text{A.2})$$

From this expression, it is obvious that the mass of the  $n$ -th KK W-boson field  $W_\mu^{(n)}$  and  $\tilde{W}_\mu^{(n)}$  is given by  $(m^{(n)})^2 = M_W^2 + n^2 M_{KK}^2$ . When we substitute the expansion (A.2) to the coupling

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<sup>¶</sup>We neglect the component  $\mu = \tau$  of the W-boson for simplicity. It is just equivalent to an adjoint scalar field, with which the computations presented below will follow in the same manner.

(A.1), we obtain two couplings

$$\frac{g}{2} \sum_n \int d^4x \bar{q}_L \gamma^\mu \tilde{W}_\mu^{(n)} Q_L, \quad \frac{g}{2} \sum_n (-1)^n \int d^4x \bar{Q}_R \gamma^\mu \tilde{W}_\mu^{(n)} q_R. \quad (\text{A.3})$$

In the latter coupling, the factor  $(-1)^n$  comes from  $\cos(\pi n)$  which indicates that this coupling resides on the point  $\tau = \pi/M_{\text{KK}}$ .

Next, from these couplings, we integrate out the massive W-bosons. By Wick-contracting the  $\tilde{W}$  field in the couplings (A.3), we obtain a four-Fermi coupling,

$$\begin{aligned} & -2 \int d^4x \int d^4y \int \frac{d^4p}{(2\pi)^4} \left[ \frac{M_{\text{KK}}}{2\pi} \frac{1}{p^2 + M_W^2} + \frac{M_{\text{KK}}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{p^2 + M_W^2 + n^2 M_{\text{KK}}^2} \right] \\ & \times e^{ip \cdot (x-y)} \times g^2 \bar{q}_L(x) q_R(y) \bar{Q}_R(x) Q_L(y). \end{aligned} \quad (\text{A.4})$$

We have used a Fierz identity. Note that the first factor  $M_{\text{KK}}/(2\pi)$  comes as a normalization of the kinetic term of the zero mode in the KK expansion,  $\int d\tau 1^2 = 2\pi/M_{\text{KK}}$ , while the factor  $M_{\text{KK}}$  in the second term comes similarly as  $\int_0^{2\pi/M_{\text{KK}}} d\tau \cos^2(n\tau M_{\text{KK}}) = \pi/M_{\text{KK}}$ . In (A.4), let us assume a large  $M_W$ , so the momentum can be neglected,  $-\partial^2 \equiv p^2 \ll M_W^2$ . Then we get the expression

$$-2 \left[ \frac{M_{\text{KK}}}{2\pi} \frac{1}{M_W^2} + \frac{M_{\text{KK}}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{M_W^2 + n^2 M_{\text{KK}}^2} \right] \int d^4x g^2 \bar{q}_L(x) q_R(x) \bar{Q}_R(x) Q_L(x). \quad (\text{A.5})$$

Using the formula

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + s^2} = -\frac{1}{2s^2} + \frac{\pi}{2s} \frac{1}{\sinh s\pi}, \quad (\text{A.6})$$

The four-Fermi coefficient is evaluated as

$$-g^2 \frac{1}{M_W} \frac{1}{\sinh(\pi M_W/M_{\text{KK}})} \int d^4x \bar{q}_L(x) q_R(x) \bar{Q}_R(x) Q_L(x). \quad (\text{A.7})$$

When the techni-quarks  $Q$  on the D4'-branes get condensed to form  $\langle \bar{Q}_L Q_R \rangle$ , this four-Fermi term generates a mass term for the quarks. In our approximation, the effectiveness requires  $M_W \gg M_{\text{KK}}$ . In this parameter region, the quark mass is obtained as<sup>||</sup>

$$m_q = \frac{g^2 \langle \bar{Q}_R Q_L \rangle}{2M_W} \exp \left[ -\frac{\pi M_W}{M_{\text{KK}}} \right] = \frac{\pi g_{\text{YM}}^2 \langle \bar{Q}_R Q_L \rangle}{M_{\text{KK}} M_W} \exp \left[ -\frac{\pi M_W}{M_{\text{KK}}} \right]. \quad (\text{A.8})$$

The exponential factor is important, it coincides with our worldsheet instanton calculation.

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<sup>||</sup> In the evaluation we drop trivial overall numerical factors.

## B Curved shape of the probe D6-brane

In the flat spacetime background, the probe D6-brane is flat. But once we replace the  $N$  D4-branes by its geometry, the worldvolume of the D6-brane is curved due to the force between the D6-branes and the D4-branes, so that it is stable. Let us calculate this stable configuration. We introduce spherical coordinates for the directions transverse to the D4-brane,

$$x^5 = U \cos \theta_1, \quad x^6 = U \sin \theta_1 \cos \theta_2, \dots \quad (\text{B.1})$$

To avoid difficulties in computation, we make an approximation that the D6-brane configuration is almost the same as that of the D6-brane not ending on the D8-brane but wrapping the  $x^4$  circle. We choose the following ansatz for the D6-brane worldvolume parameterized by  $(\sigma_1, \sigma_2, x^{0,1,2,3,4})$ ,

$$\theta_3 = \theta_4 = 0, \quad \theta_2 = \sigma_2, \quad \theta_1 = \sigma_1, \quad U = U(\sigma_1) \quad (\text{B.2})$$

which is maximally rotation-symmetric. The D6-brane effective action, which is a 7-dimensional DBI action in the Witten's background spacetime, is given by

$$S = \mathcal{T}_{\text{D6}} \int d\sigma_1 d\sigma_2 d^5 x e^{-\phi} \sqrt{-\det g} = V \int d\theta_1 U^{5/2} \sin \theta_1 \sqrt{\left(\frac{dU}{d\theta_1}\right)^2 + U^2 f(U)}, \quad (\text{B.3})$$

where  $V = R^{-3/2} \mathcal{T}_{\text{D6}} \int d^5 x$ . We can easily solve the equation of motion of this action numerically, with the initial condition

$$U(\theta_1 = 0) = U_0, \quad \left. \frac{dU}{d\theta_1} \right|_{\theta_1=0} = 0. \quad (\text{B.4})$$

Here  $U_0$  is a constant parameter. This  $U_0$  is the minimum distance between the D6-brane and the D4-branes. Our numerical result shows a consistent configuration of a curved D6-brane in the background. The worldvolume point  $\theta_1 = 0$  is the closest to the D4-branes. It is the same as the flat configuration of the D6-brane in the flat background.

## C Two-instantons and $\pi^0$ - $\pi^\pm$ mass difference

The two instanton sector is described in the following way. The instanton number is just the wrapping number of the boundary of the worldsheet on the non-trivial cycle on the D8-brane. When it winds once, the instanton contribution is proportional to (3.10). So, the two instanton sector is given by

$$\sum_i \sum_j \text{tr} \left[ \text{P exp} \left[ -\Phi \int_b dz \right] \cdot \text{P exp} \left[ -i \oint_i A_z dz \right] \cdot \text{P exp} \left[ -\Phi \int_b dz \right] \cdot \text{P exp} \left[ -i \oint_j A_z dz \right] \right] \quad (\text{C.1})$$

Here, again we have followed the approximation that the integral regions of  $\Phi$  is separated from the integral region of  $A_z$ . Then, using (3.11) and looking at the front factor in the one-instanton result (3.12), we obtain

$$S_{2\text{-instanton}} = \frac{1}{3^{9/2}\pi^3} g_{\text{YM}}^3 N_c^{3/2} M_{\text{KK}}^3 \mathcal{N}^{-2} \cdot \frac{g_{\text{YM}}^2}{2\pi M_{\text{KK}}} \int d^4x \text{tr} [m_q U m_q U + U^\dagger m_q U^\dagger m_q] \quad (\text{C.2})$$

This expression is, again, consistent with chiral perturbation theory. We expand this expression with  $U = \exp[2i\pi(x)/f_\pi]$ , then we obtain terms quadratic in the pion fields as

$$\begin{aligned} \text{tr} [m_q U m_q U + U^\dagger m_q U^\dagger m_q]_{\mathcal{O}(\pi^2)} &= -\frac{8}{f_\pi^2} \text{tr} [m_q \pi(x) m_q \pi(x) + m_q^2 \pi(x)^2] \\ &= -\frac{8}{f_\pi^2} \left( (m_u^2 + m_d^2) (\pi^0(x))^2 + \frac{1}{2} (m_u + m_d)^2 ((\pi^+(x))^2 + (\pi^-(x))^2) \right) \end{aligned}$$

where  $\pi^\pm \equiv (\pi_1 \pm i\pi_2)/\sqrt{2}$  and  $\pi^0 = \pi_3$ . From this, the mass difference is obtained as

$$|m_{\pi^\pm}^2 - m_{\pi^0}^2| = \frac{2}{3\sqrt{3}} \frac{g_{\text{YM}}^3}{\sqrt{N_c} \mathcal{N}^2} (m_u - m_d)^2. \quad (\text{C.3})$$

Let us give a numerical estimate as an illustration. If we substitute  $N_c = 3$ ,  $g_{\text{YM}} = 2.35$  and also the experiment values of the pion masses,  $m_{\pi^\pm} = 139.6$  [MeV],  $m_{\pi^0} = 135.0$  [MeV], then with  $\mathcal{N} = 1$ , we obtain  $|m_u - m_d| \simeq 21$  [MeV]. This is larger than the experimental values of the quark masses.

The origin of this discrepancy can be understood as follows. In a chiral perturbation theory, there are other higher order terms,  $(\text{tr}[m_q U])^2$  and  $(\text{tr}[m_q U^\dagger])^2$ . Together with these terms, the realistic pion mass difference is reproduced. In our holographic QCD approach, these double-trace operators appear at a sub-leading order in the large  $N$  expansion and so does not appear in our leading-order estimates. It would be interesting to compute these sub-leading corrections and see how the masses of  $K^{0,\pm}$  can be reproduced from strange quark mass.

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